ARITHMETIC DIFFERENTIAL EQUATIONS OF PAINLEVÉ VI TYPE

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ABSTRACT. *Using the description of Painlevé VI family of differential equations in terms of a universal elliptic curve, going back to R. Fuchs (cf. [Ma96]), we translate it into the realm of Arithmetic Differential Equations (cf. [Bu05]), where the role of derivative "in the p–adic direction" is played by a version of Fermat quotient.*

Introduction and brief summary

This article is dedicated to the study of differential equations of the Painlevé VI type "with p–adic time", in which the role of time derivative is played by a version of p–Fermat quotient. Richness of p–adic differential geometry was already demonstrated in many papers of the first–named author: see in particular the monograph [Bu05].

Applicability of this technique to the Painlevé equations is ensured by the combination of two approaches: R. Fuchs's treatment of the classical case modernised in [Ma96], and constructions of p–adic differential characters in [Bu95].

In sec. 1 below we present a short introduction to the p–adic differential geometry. In sec. 2 and 3 we introduce several versions of p–adic PVI. Sec. 4 is dedicated to the problem of transposing into p–adic domain of main features of Hamiltonian formalism. We have not found a definitive answer to this problem, and Painlevé equations serve here as a very stimulating testing ground. Finally, section 5 treats another problem, suggested by comparison of several versions of p–adic PVI, but presenting an independent interest.

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1. Arithmetic differential equations: background

We start with a brief summary of relevant material from [Bu05], Chapters 2, 3. Fix a prime p; in our applications it will be assumed that $p \geq 5$.

1.1. p**–derivations.** Recall that, in the conventional commutative algebra, given a ring A and an A–module N, *a derivation of* A *with values in* N is any map $\partial : A \to N$ such that the map $A \to A \times N : a \mapsto (a, \partial a)$ is a ring homomorphism, where $A \times N$ is endowed with the structure of commutative ring with componentwise addition, and multiplication $(a, m) \cdot (b, n) := (ab, an + bm)$. Notice that $\{0\} \times N$ is the ideal with square zero in $A \times N$.

Similarly, in arithmetic geometry *a* p*–derivation of* A *with values in an* A*–algebra* $B,\ f : A \to B,$ is a map $\delta_p : A \to B$ such that the map $A \to B \times B : a \mapsto (f(a), \delta_p(a))$ is a ring homomorphism $A \to W_2(B)$ where $W_2(B)$ is the ring of p–typical Witt vectors of length 2. Again, if $pB = \{0\}$, Witt vectors of the form $(0, b)$ form the ideal of square zero.

Explicitly, this means that $\delta_p(1) = 0$, and

$$\delta_p(x + y) = \delta_p(x) + \delta_p(y) + C_p(x, y), \tag{1.1}$$

$$\delta_p(xy) = f(x)^p \cdot \delta_p(y) + f(y)^p \cdot \delta_p(x) + p \cdot \delta_p(x) \cdot \delta_p(y), \tag{1.2}$$

where

$$C_p(X, Y) := \frac{X^p + Y^p - (X + Y)^p}{p} \in \mathbf{Z}[X, Y]. \tag{1.3}$$

In particular, this implies that for any p–derivation $\delta_p : A \to B$ the respective map $\phi_p : A \to B$ defined by $\phi_p(a) := f(a)^p + p\delta_p(a)$ is a ring homomorphism satisfying $\phi_p(x) \equiv f(x)^p \bmod p$, that is "a lift of the Frobenius map applied to f".

Conversely, having such a lift of Frobenius, we can uniquely reconstruct the respective derivation δ_p *under the condition that* B *has no* p*–torsion.*

We will often work with p–derivations $A \to A$ with respect to the identity map $A \to A$ and keep p fixed; then it might be kept off the notation. Such a pair (A, δ) is called a δ–ring. Morphisms of δ–rings are algebra morphisms compatible with their p–derivations.

Moreover, our rings (and more generally, schemes) will be R–algebras where $R = W(k)$ (ring of infinite p–typical Witt vectors) is the completion of the maximal unramified extension of $\mathbf{Z}_p$, with residue field $k :=$ an algebraic closure of $\mathbf{Z}/p\mathbf{Z}$.

By $\phi: R \to R$ we denote the automorphism acting as Frobenius $x \mapsto x^p$ on k, and by δ the respective p–derivation: $\delta(x) = (\phi(x) - x^p)/p$. The R–algebra structure on a δ–ring is always assumed to be compatible with this p-derivation.

1.2. Prolongation sequences and p–jet spaces. In the classical situation invoked in 1.1, there exists an universal derivation

$$d: A \to \Omega^1(A) \tag{1.4}$$

with values in the A–module of differentials.

For p–derivations, (1.4) might be replaced by the following construction (however, see the Remark 1.2.1 below).

Let A be an R–algebra. *A prolongation sequence* for A consists of a family of p–adically complete R–algebras $A^i, i \geq 0$, where $A^0 = A^\frown$ is the p–adic completion of A, and of maps $\varphi_i, \delta_i: A^i \to A^{i+1}$ satisfying the following conditions:

a) φ_i are ring homomorphisms, each δ_i is a p–derivation with respect to φ_i, compatible with δ on R.

b) $\delta_i \circ \varphi_{i-1} = \varphi_i \circ \delta_{i-1}$ for all $i \geq 1$.

Prolongation sequences form a category with evident morphisms, ring homomorphisms $f_i : A^i \to B^i$ commuting with φ_i and δ_i, and in its subcategory with fixed A^0 *there exists an initial element, defined up to unique isomorphism* (cf. [Bu05], Chapter 3). It can be called the universal prolongation sequence.

In the geometric language, if $X = \operatorname{Spec} A$, the formal spectrum of the i–th ring A^i in the universal prolongation sequence is denoted $J^i(X)$ and called *the i–th p–jet space of X*. Conversely, $A^i = \mathcal{O}(J^i(X))$, the ring of global functions.

The geometric morphisms (of formal schemes over $\mathbf{Z}$) corresponding to ϕ_i are denoted $\phi^i: J^i(X) \to J^0(X) =: X^\frown$(formal p–adic completion of X).

This construction is compatible with localisation so that it can be applied to the non–necessarily affine shemes: cf. [Bu05], Chapter 3.

1.2.1. Remark. Classically, (1.4) extends to the universal map of A to the differential graded algebra $\Omega^*(A)$, and there is a superficial similarity of this map with the one, say of A to the inductive limit of its universal prolongation sequence in the p–adic arithmetics context.

However, the classical differential acts in $\mathbf{Z}_2$–graded supercommutative algebras and is an *odd* operator with square zero, whereas δ_p are even.

The differential geometry of smooth schemes *in characteristic* $p > 0$ suggests a perspective worth exploring. Namely, the sheaf of differential forms on such a scheme is endowed with the so called *Cartier operator* C, which is dual to the Frobenius operator $F : \partial \mapsto \partial^p$ acting upon vector fields. This operator C is F^{-1}–linear. One could consider studying p–adic lifts of the Cartier operator from the closed fibre of the relevant scheme to its p–adic completion, following the lead of [Bu05]. For a recent survey of F^{-1}–linear maps cf. [BlSch12] and references therein.

1.3. Flows. Let now X be a smooth affine scheme over $R = W(k)$. Each element of $\mathcal{O}(J^r(X))$ induces a function $f : X(R) \mapsto R$. Such functions are called δ–functions of order r on X, and we may and will identify them with respective elements of $\mathcal{O}(J^r(X))$. For $r = 0$, $\mathcal{O}(J^0(X)) = \mathcal{O}(X)\widehat{}$, the p–adic completion of $\mathcal{O}(X)$.

1.3.1. Definition. *a) A system of arithmetic differential equations of order* r *on* X *is a subset* $\mathcal{E}$ *of* $\mathcal{O}(J^r(X))$.

b) A solution of $\mathcal{E}$ *is an* R*–point* $P \in X(R)$ *such that* $f(P) = 0$ *for all* $f \in \mathcal{E}$. *The set of solutions of* $\mathcal{E}$ *is denoted* $Sol(\mathcal{E}) \subset X(R)$.

c) A prime integral of $\mathcal{E}$ *is a function* $\mathcal{H} \in \mathcal{O}(X)\widehat{}$ *such that* $\delta(\mathcal{H}(P)) = 0$ *for all* $P \in Sol(\mathcal{E})$.

We will also denote by $Z^r(\mathcal{E})$ the closed formal subscheme of $J^r(X)$ generated by $\mathcal{E}$.

Now, let δ_X be a p–derivation of $\mathcal{O}(X)\widehat{}$. From the universality of the jet sequence explained in 1.2, it follows that such derivations are in a bijection with the sections of the canonical morphism $J^1(X) \to J^0(X)$.

1.3.2. Definition. *The* δ*–flow associated to* δ_X *is the system of arithmetic differential equations of order 1 which is the ideal in* $\mathcal{O}(J^1(X))$ *generated by elements of the form* $\delta f_i - \delta_X f_i$ *where* $f_i \in \mathcal{O}(X)$ *generate* $\mathcal{O}(X)$ *as* R*–algebra.*

We use the word "flow" in this context in order to suggest that in our main applications we consider the p–adic axis as an arithmetic version of the time axis.

The derivation δ_X is completely determined by its δ–flow. If δ_X corresponds to the section $s : J^0(X) \to J^1(X)$ of $J^1(X) \to J^0(X)$ then $Z(\mathcal{E}(\delta_X)) \subset J^1(X)$ coincides with the image of ths section. One easily checks that if $\mathcal{H} \in \mathcal{O}(X)\widehat{}$ is such that $\delta_X\mathcal{H} = 0$ then $\mathcal{H}$ is a prime integral for $\mathcal{E}(\delta_X)$. All of the above can be transposed to the case when X a p–formal scheme, locally a p–adic completion of a smooth scheme over R.

In what follows we choose a smooth affine scheme Y and apply the constructions discussed above to $X := J^1(Y)$. In this case one can define a special class of δ-flows on $J^1(Y)$ which will be called *canonical δ-flows.*

1.3.3. Definition. *A canonical δ–flow is a δ–flow $\mathcal{E}(\delta_{J^1(Y)})$ with the property that the composition of $\delta_{J^1(Y)} : \mathcal{O}(J^1(Y)) \to \mathcal{O}(J^1(Y))$ with the pull back map $\mathcal{O}(Y) \to \mathcal{O}(J^1(Y))$ equals the universal p–derivation $\delta : \mathcal{O}(Y) \to \mathcal{O}(J^1(Y))$.*

Notice that in view of the universality property of p–jet spaces, one gets a natural closed embedding $\iota : J^2(Y) \to J^1(J^1(Y))$. This induces an injective map (which we view as an identification) from the set of sections of $J^2(Y) \to J^1(Y)$ to the set of sections of $J^1(J^1(Y)) \to J^1(Y)$. The sections of $J^2(Y) \to J^1(Y)$ are in a natural bijection with canonical δ–flows on $J^1(Y)$ whereas the sections of $J^1(J^1(Y)) \to J^1(Y)$ are in a bijection with (not necessarily canonical) δ–flows on $J^1(Y)$.

Finally, consider a system of arithmetic differential equations of order 2, $\mathcal{F} \subset \mathcal{O}(J^2(Y))$.

1.3.4. Definition. *$\mathcal{F}$ defines a δ–flow on $J^1(Y)$ if the map $Z^2(\mathcal{F}) \to J^1(Y)$ is an isomorphism.*

In this case then $Z^2(\mathcal{F}) \to J^1(Y)$ defines a section of $J^2(Y) \to J^1(Y)$ and hence a canonical δ–flow $\mathcal{E}(\delta_{J^1(Y)})$ on $J^1(Y)$ such that $\iota(Z^2(\mathcal{F})) = Z^1(\mathcal{E}(\delta_{J^1(Y)}))$.

The differential algebra counterpart of the above definition yields the natural concept of flow on the (co)tangent space defining a second order differential equation.

2. Painlevé VI and differential characters of elliptic curves

2.1. Classical case. The family of sixth Painlevé equations depends on four arbitrary constants $(\alpha, \beta, \gamma, \delta)$ and is classically written as

$$\frac{d^2X}{dt^2} = \frac{1}{2}\left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-t}\right)\left(\frac{dX}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{X-t}\right)\frac{dX}{dt} +$$

$$+\frac{X(X-1)(X-t)}{t^2(t-1)^2}\left[\alpha + \beta\frac{t}{X^2} + \gamma\frac{t-1}{(X-1)^2} + \delta\frac{t(t-1)}{(X-t)^2}\right]. \tag{2.1}$$

As R. Fuchs remarked in 1907, (2.1) can be rewritten as the differenttial equation for a (local) section $P := (X(t), Y(t))$ of the generic elliptic curve $E = E(t):\ Y^2 = X(X-1)(X-t)$:

$$t(1-t)\left[t(1-t)\frac{d^2}{dt^2} + (1-2t)\frac{d}{dt} - \frac{1}{4}\right]\int_{\infty}^{(X,Y)} \frac{dx}{\sqrt{x(x-1)(x-t)}} =$$

$$= \alpha Y + \beta\frac{tY}{X^2} + \gamma\frac{(t-1)Y}{(X-1)^2} + (\delta - \frac{1}{2})\frac{t(t-1)Y}{(X-t)^2} \tag{2.2}$$

The l. h. s. of (2.2) can be called *the additive differential character μ of order two* of E: it is a non–linear differential expression in coordinates of P such that $\mu(P+Q) = \mu(P) + \mu(Q)$ where $P+Q$ means addition of points of the generic elliptic curve E, with infinity as zero. In particular, $\mu(Q) = 0$ for points of finite order. The point is that the integral in the l.h.s. of (2.2) already has such additivity property, but it is defined only modulo periods, and the latter are annihilated by the Gauss differential operator.

Thus $\mu(P)$ is defined up to multiplication by an invertible function of t. If we choose a differential of the first kind ω on the generic curve and the symbol of the Picard–Fuchs operator of the second order annihilating periods of ω, the character will be defined uniquely. In particular, if we pass to the analytic picture replacing the algebraic family of curves $E(t)$ by the analytic one $E_\tau := \mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau) \mapsto \tau \in H$, and denote by z a fixed coordinate on $\mathbf{C}$, then (2.1) and (2.2) can be equivalently written in the form

$$\frac{d^2 z}{d\tau^2} = \frac{1}{(2\pi i)^2}\sum_{j=0}^{3}\alpha_j \wp_z(z + \frac{T_j}{2}, \tau) \tag{2.3}$$

where $(\alpha_0, \dots, \alpha_3) := (\alpha, -\beta, \gamma, \frac{1}{2} - \delta)$ and

$$\wp(z,\tau) := \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)}\left(\frac{1}{(z+m\tau+n)^2} - \frac{1}{(m\tau+n)^2}\right). \tag{2.4}$$

Moreover, we have

$$\wp_z(z,\tau)^2 = 4(\wp(z,\tau) - e_1(\tau))(\wp(z,\tau) - e_2(\tau))(\wp(z,\tau) - e_3(\tau)) \tag{2.5}$$

where

$$e_i(\tau) = \wp(\frac{T_i}{2}, \tau),\ (T_0, \dots, T_3) = (0, 1, \tau, 1+\tau) \tag{2.6}$$

so that $e_1 + e_2 + e_3 = 0$.

For a more geometric description, cf. sec. 4.5 below.

2.2. p–adic differential additive characters. The form (2.3) is more suggestive than (2.1) for devising a p–adic version of PVI. In purely algebraic terms, z can be described as the logarithm of the formal group law, and the rhs of (2.3) is simply a linear combination of shifts of $\wp_z$- (or Y-)coordinate by sections of the second order.

More precisely, using basic conventions of [Bu05], let $p \geq 5$ be a prime, k an algebraic closure of $\mathbf{Z}/p\mathbf{Z}$, $R = W(k)$ the ring of p–typical Witt vectors., as in 1.1.

Consider a smooth projective curve of genus one E over R, with four marked and numbered R–sections P_i, $i = 0, \dots, 3$, such that all divisors $2(P_i - P_j)$ are principal ones. Choosing, say, P_0 as zero section, we may and will identify E with its Jacobian and represent E as the closure of the affine curve $y^2 = 4x^3 + ax + b = 4(x - e_1)(x - e_2)(x - e_3)$, with $e_i \in R$, corresponding to $P_i - P_0$. Assume that E is *not a canonical lift of its (good) reduction*, that is, does not admit a lift of the Frobenius morphism of $E \otimes k$.

Choose the differential $\omega := dx/y$. Whenever we work with several elliptic curves simultaneously, we will write $x_{E,\omega}, y_{E,\omega}$ in place of former x, y etc.

The curve E has a canonical p–differential character $\psi_{E,\omega}$ of order 2 ([Bu05], pp. 201 and 197), which corresponds to $(2\pi i)^2 d^2 z/d\tau^2$ in (2.3).

2.3. Painlevé p–adic equation and the problem of constants. Now we can directly write a p–adic version of (2.3) as

$$\psi_{E,\omega}(Q) = \sum_{j=0}^{3} \alpha_j s_j^*(y(Q)) \tag{2.7}$$

where Q is a variable section of E/R, and $s_j : E_j \to E_j$ is the shift by P_j.

At this point we have to mention two problems.

(A) In (2.3), α_j must be absolute constants rather than, say, functions of t. Directly imitating this condition, we have to postulate that in (2.7), α_j must be roots of unity or zero, i. e. δ_p–constants. It is desirable to find a justification of such a requirement (or a version of it) in a more extended p–adic theory of Painlevé VI, e. g. tracing its source to the arithmetic analog of isomonodromy deformations.

(B) The relevance and non–triviality of the problem of "constants" in the p–adic differential equations context is also implicit in our exclusion of those E that are canonical lifts of their reductions.

A formal reason for this exclusion was the fact for such E the basic differential character ψ has order 1 rather than 2, thus being outside the framework of Painlevé VI. But the analogy with functional case suggests that canonical lifts should be morally considered as analogs of families with constant j–invariant in the functional case.

This agrees also with J. Borger's philosophy that Frobenius lift(s) should be treated as descent data to an *"algebraic geometry below* Spec $\mathbf{Z}$*"* (cf. [Bo05]). In our particular case the latter might be called "geometry over p–typical field with one element" (Borger's suggestion in e-mail to Yu.M. of April 22, 2013).

Indeed, canonical lifts X in such a geometry are endowed with an isomorphism $X^{\phi} \to X^{(p)}$ that can be seen as a categorification of the identity $c^{\phi} = c^p$ defining roots of unity.

In the p–adic case, however, if we decide to declare j–invariants of canonical lifts "constants" in some sense, this will require a revision of the latter notion. These invariants generally are not roots of unity: cf. Finotti's papers in http://www.math.utk.edu/~finotti/ .

3. Symmetries and variants

3.1. Lemma. *(i) (Landin's transform). In the notations of 2.3, denote for each* $i = 1, 2, 3$ *by* $\pi_i : E \to E_i := E/\langle P_i \rangle$ *the respective isogeny. Let* ω_i *be the 1–form on* E_i *such that* $\pi^*(\omega_i) = \omega$*. Then*

$$\pi_i^*(y_{E_i,\omega_i}) = y_{E,\omega} + s_i^*(y_{E,\omega}). \tag{3.1}$$

(ii) We have

$$\pi_i^*(\psi_{E_i,\omega_i}) = \psi_{E,\omega}. \tag{3.2}$$

Proof. (i) If we choose an embedding of R in $\mathbf{C}$ identifying $E(\mathbf{C})$ with $\mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$ in such a way that P_i becomes the point $\tau/2$ (modulo periods), then (3.1) turns into the classical Landin identity (cf. [Ma96], sec. 1.6):

$$\wp(z,\tau/2)=\wp(z,\tau)+\wp(z+\tau/2,\tau)$$

(ii) The identity (3.2) can be stated and proved in wider generality. Namely, let E, E' be two smooth elliptic curves over R, not admitting lifts of Frobenius. Let $\pi : E \to E'$ be an isogeny of degree prime to p and and ω, ω' such bases of 1–forms on E, E' that $\pi^*(\omega')=\omega$. Then

$$\pi^*(\psi_{E',\omega'})=\psi_{E,\omega}. \tag{3.3}$$

In fact from [Bu05], Theorem 7.34, it follows that $\pi^*(\psi_{E',\omega'}) = c\psi_{E,\omega}$ for some $c \in R$.

Now consider the second jet space $J^2(E)$. Then $\psi = \psi_{E,\omega}$ is an element of $\mathcal{O}(J^2(E))$ (cf. [Bu05], p. 201). Denote by $\phi^i : J^i(E) \to J^0(E)$ the map defined at the end of sec. 1.2. Put

$$\omega^{(i)} := \frac{(\phi^i)^*(\omega)}{p^i},\ i=0,1,2. \tag{3.4}$$

Then in view of [Bu05], p. 203 (where our $\omega^{(i)}$ were denoted ω_i), we have

$$d\psi = p\omega^{(2)}+\lambda_1\omega^{(1)}+\lambda_0\omega^{(0)} \tag{3.5}$$

with $\lambda_1 \in R$, $\lambda_1 \in R^\times$ depending on (E,ω). Notice that here d means the usual differential taken in the "geometric", or "vertical" direction, that is $dc=0$ for any $c \in R$. Moreover, in [Bu97] it was proved that if E is defined over $\mathbf{Z}_p \subset R$, then $\lambda_0 = 1$, $\lambda_1 = -a_p$ where a_p is the trace of Frobenius on the reduction $E \bmod p$.

Returning to the proof of (3.3), we can now compare the $\omega^{(2)}$–contributions to $d\psi_{E',\omega'}$ in two different ways. First, we have

$$\pi^*(d\psi_{E',\omega'})=\pi^*(p(\omega')^{(2)}+\dots)=p\,(\pi^*(\omega')^{(2)})+\dots$$

Second,

$$\pi^*(d\psi_{E',\omega'})=d\pi^*(\psi_{E',\omega'})=d(c\,\psi_{E,\omega})=c\ (p\omega^{(2)}+\dots).$$

This shows that $c=1$.

3.2. Two versions of PVI. Put $Y := E \setminus \cup_{i=0}^{3} P_i$. Denote by $r := \sum_{j=0}^{3} \alpha_j s_j^*(y) \in \mathcal{O}(Y)$ (cf. (2.7)).

Below we will denote by ρ *either* r*, or* $\phi(r) \in \mathcal{O}(J^1(Y))$. Although the introduction of the version with $\phi(r)$ is not motivated at this point, we will see in sec. 4 below that exactly this version admits a "p–adic Hamiltonian" description.

The character ψ induces the map of sets, that we will also denote $\psi : E(R) \to R$; similarly, ρ induces the map of sets $\rho : Y(R) \to R$.

3.2.1. Proposition. *Denote by the symbol* $PVI(E, \omega, P_0, P_1, P_2, P_3, \alpha_0, \alpha_1, \alpha_2, \alpha_3)$ *any of the two equations*

$$\psi(P) = \rho(P). \tag{3.6}$$

Let $\pi := \pi_2 : E \to E' := E/\langle P_2 \rangle$ *as in Lemma 3.1. Put* $P_0' = \pi(P_0)$ *(zero point),* $P_1' = \pi(P_1)$*, and choose remaining two points of order two* P_2', P_3'*. Assume that* $\pi^*(\omega') = \omega$*.*

In this case for any solution Q *to (3.6), the point* $Q' := \pi(Q)$ *will be a solution to*

$$PVI' = PVI(E', \omega', P_0', P_1', P_2', P_3', \alpha_0, \alpha_1, 0, 0), \tag{3.7}$$

and conversely, if Q' *is a solution to* PVI'*, then* Q *is a solution to* PVI*.*

Proof. Let $\rho = \phi^j(r)$, $j = 0$ or 1. Our statement results from the following calculation, using (3.1), (3.2):

$$\psi_{E',\omega'}(\pi(Q)) = \psi_{E,\omega}(Q) =$$

$$\phi^j(\alpha_0(y_{E,\omega}(Q) + y_{E,\omega}(Q + P_2)) + \alpha_1(y_{E,\omega}(Q + P_1) + y_{E,\omega}(Q + P_1 + P_2))) =$$

$$\phi^j(\alpha_0 y_{E',\omega'}(\pi(Q)) + \alpha_1 y_{E',\omega'}(\pi(Q) + \pi(P_1))).$$

Of course, similar statement will hold if π_2 is replaced by π_1 or π_3.

3.3. Two more versions of PVI. The character ψ is of course not algebraic, but the equation (3.5) shows that it has an algebraic vertical differential. The same is obviously true for the rhs of (2.7): one easily sees that if E is given in the Weierstrass form $y^2 = f(x)$, then

$$dr = -\frac{1}{2} \sum_{j=0}^{3} \alpha_j s_j^*(f'(x)))\,\omega. \tag{3.8}$$

Hence we may consider two more versions of the arithmetic PVI: the condition of vanishing of one of the following 1–forms on $J^2(E)$:

$$d\psi - dr = p\omega^{(2)} + \lambda_1\omega^{(1)} + \left(\lambda_0 - \frac{dr}{\omega}\right)\omega^{(0)}, \tag{3.9a}$$

$$d\psi - d\phi(r) = p\omega^{(2)} + \left(\lambda_1 - p\phi\left(\frac{dr}{\omega}\right)\right)\omega^{(0)} + \lambda_0\omega^{(0)}. \tag{3.9b}$$

But in the p–adic situation solutions to (3.6) and (3.9a,b) respectively are not related to each other in the way we would expect by analogy with usual calculus.

Indeed, let us consider a smooth function $f(x, y, y', ..., y^{(r)})$ in $r+2$ variables defined on $\mathbf{R}^{r+2}$, where the latter is viewed as the r–th jet space of the first projection $\mathbf{R}^2 \to \mathbf{R}$, $(x, y) \mapsto x$. Let $u = u(x)$ be an unknown smooth function $u : \mathbf{R} \to \mathbf{R}$. Then the equation

$$f(x, u(x), u'(x), ..., u^{(r)}(x)) = 0 \tag{3.10}$$

is related to the 1-form df on $\mathbf{R}^{r+2}$ as follows. If u solves (3.10), then taking the derivative of (3.10) with respect to x we get

$$\nabla^r(u)^*(df) = 0 \tag{3.11}$$

where $\nabla^r(u) : \mathbf{R} \to \mathbf{R}^{r+2}$ is given by $\nabla^r(u)(x) = (x, u(x), ..., u^{(r)}(x))$.

However in the case of arithmetic jet spaces the situation is different.

Indeed, let $f \in \mathcal{O}(J^r(\mathbf{A}^1)) = R[y, y', ..., y^{(r)}]\hat{}$ where $J^r(\mathbf{A}^1)$ is the arithmetic jet space of the affine line over R. Furthermore, let $u \in R$ be a solution of

$$f(u, \delta u, ..., \delta^r u) = 0 \tag{3.12}$$

where $\delta = \delta_p : R \to R$ is the standard p–derivation on R. Applying δ to (3.12) we get:

$$\frac{1}{p}(f^{(\phi)}(u^p + p\delta u, (\delta u)^p + p\delta^2 u, ..., (\delta^r u)^p + p\delta^{r+1}u) - f(u, \delta u, ..., \delta^r u)^p) = 0 \tag{3.13}$$

where $f^{(\phi)}$ is f with coefficients twisted by the Frobenius ϕ. To compare this latter equation with (3.11), one can apply the Taylor formula and get

$$\frac{f^{(\phi)}(\nabla^r(u)^p) - f(\nabla^r(u))^p}{p} + \sum_{i=0}^{r}\frac{\partial f^{(\phi)}}{\partial y^{(i)}}(\nabla^r(u)^p)(\delta^{i+1}u) + M = 0, \tag{3.14}$$

where $\nabla^r(u) = (u, \delta u, ..., \delta^r u)$.

The first term here is an arithmetic version of the pullback of the term $\frac{\partial f}{\partial x}dx$ in df. The second term clearly involves df. However M involves higher partial derivatives of f; it is highly non-linear, it is divisible by p, and "the more non-linear its terms are the more they are divisible by p". In some sense, since p is small, one can view (3.14) as a non-linear deformation of (3.11).

We will continue discussion of this formalism in sec. 5 below. The reader may wish to skip the next section, or to postpone reading it.

4. Hamiltonian formalism

4.1. Classical case. The classical PVI equation written in the form (2.3) can be represented as a Hamiltonian flow on the variable two–dimensional phase space (twisted cotangent spaces to a versal family of elliptic curves, cf. [Ma96] and the end of this section), with time–dependent Hamiltonian:

$$\frac{dz}{d\tau} = \frac{\partial \mathcal{H}}{\partial y}, \ \frac{dy}{d\tau} = -\frac{\partial \mathcal{H}}{\partial z}, \tag{4.1}$$

where

$$\mathcal{H} := \mathcal{H}(\alpha_0, \dots, \alpha_3) := \frac{y^2}{2} - \frac{1}{(2\pi i)^2} \sum_{j=0}^{3} \alpha_j \wp(z + \frac{T_j}{2}, \tau). \tag{4.2}$$

In more geometric terms, this means that solutions to the PVI become leaves of the null–foliation of the following closed two–form:

$$\omega = \omega(\alpha_0, \dots, \alpha_3) := 2\pi i(dy \wedge dz - d\mathcal{H} \wedge d\tau) =$$

$$= 2\pi i(dy \wedge dz - y dy \wedge d\tau) + \frac{1}{2\pi i} \sum_{j=0}^{3} \alpha_j \wp_z(z + \frac{T_j}{2}, \tau) dz \wedge d\tau. \tag{4.3}$$

The extra factor $2\pi i$ makes ω defined over $\mathbf{Q}[\alpha_i]$ on a natural algebraic model of (twisted) relative cotangent bundle to the respective versal family of elliptic curves.

In the expression (4.3), the summand $2\pi i\, dy \wedge dz$ is the canonical fibrewise symplectic form on the relative cotangent bundle. The terms involving $d\tau$ uniquely determine the differential of the (tme dependent) Hamiltonian.

Moreover, ω is not just closed, but is a global differential: $\omega = d\nu$ where the form

$$\nu = \nu(\alpha_0, \dots, \alpha_3) := 2\pi i\,(ydz - \frac{1}{2}y^2 d\tau) + d\log\theta(z,\tau) + 2\pi i\, G_2(\tau)d\tau +$$

$$+\frac{1}{2\pi i}\sum_{j=0}^{3}\alpha_j\wp(z+\frac{T_j}{2},\tau)d\tau \tag{4.4}$$

also descends to an appropriate algebraic model, and the Hamiltonian $\mathcal{H}$ is again encoded in the $d\tau$–part of ν. Here is a convenient way to represent this encoding:

$$\mathcal{H}(\alpha_0, \dots \alpha_3) = i_{\partial_\tau}\left(\frac{y^2}{2}d\tau + \frac{1}{2\pi i}(\nu(0,\dots,0) - \nu(\alpha_0,\dots,\alpha_3))\right), \tag{4.5}$$

where $\partial_\tau := \frac{\partial}{\partial\tau}$.

Finally, the last summand in (4.3) is simply the additive differential character $2\pi i\dfrac{d^2 z}{d\tau^2}$ that is generally denoted ψ in the p–adic case.

In the following subsections, we try to imitate this description of Hamiltonian structure for p–adic PVI equations. The reader should be aware that our treatment is somewhat *ad hoc*, and must be considered as a tentative step towards a more coherent vision of Hamiltonian flows with p–adic time. In fact, we do not yet have an appropriate version of dp replacing $d\tau$ and generally do not know what are differential forms involving "differentials in the arithmetical direction".

4.2. Arithmetical case: preparation. Let Y be an formal affine scheme over $R = W(k)$. Modules of vertical differential forms on Y are defined as

$$\Omega_Y = \lim\mathrm{inv}\,\Omega_{Y_n/R_n}$$

where $R_n = R/p^{n+1}R$, $Y_n = Y\otimes_R R_n$.

Let now $Z \subset J^n(Y)$ be a closed formal subscheme defined by the ideal $I_Z \subset \mathcal{O}(J^n(Y))$. Put

$$\Omega'_Z := \frac{\Omega_{J^n(Y)}}{\langle I_Z\Omega_{J^n(Y)}, dI_Z\rangle} \tag{4.6}$$

Given a system of arithmetic differential equations $\mathcal{F} \subset \mathcal{O}(J^r(Y))$, denote by $Z^r := Z^r(\mathcal{F})$ the ideal generated by $\mathcal{F}$. For each $s \leq r$, there is a natural map $\pi_{r,s} : Z^r \to J^s(Y)$.

Generally, the natural maps ϕ^* respect degrees of differential forms, one can define natural maps $\phi^*/p^i : \Omega^i_{J^{r-1}(Y)} \to \Omega^i_{J^r(Y)}$ and, for $f \in \mathcal{O}(J^2(Y))$, they induce maps which we will denote

$$\frac{\phi^*_Z}{p^i} : \Omega^i_{J^1(Y)} \to \Omega'^i_{Z^2(f)} \tag{4.7}$$

4.2.1. Definition. *We say that $\mathcal{F} \subset \mathcal{O}(J^r(Y))$ defines a generalised canonical δ–flow on $J^s(Y)$, if the induced map*

$$\pi^*_{r,s}\Omega_{J^s(Y)} \to \Omega'_{Z^r}$$

is injective, and its cokernel is annihilated by a power of p.

The cokernel here intuitively measures "how singular" $\mathcal{F}$ is on the closed fibre of Y.

4.2.2. Definition. *a) Let X be a smooth surface over R (or the p–adic completion of such a surface).*

A symplectic form on X is an invertible 2–form on X.

A contact form on X is an 1–form on X such that $d\nu$ is symplectic.

b) Let Y be a smooth curve over R. An 1–form on $X := J^1(Y)$ is called canonical, if $\nu = f\beta$, where $f \in \mathcal{O}(X)$ and β is an 1–form lifted from Y.

Notice that any closed canonical 1–form on $X = J^1(Y)$ is lifted from Y.

We now come to the main definition.

4.3. Definition. *Let Y be a smooth affine curve over R and let $f \in \mathcal{O}(J^2(Y))$ be a function defining a generalised canonical δ–flow on $J^1(Y)$.*

a) The respective generalized δ–flow is called Hamiltonian with respect to the symplectic form η on $J^1(Y)$, if $\phi^*_Z\eta = \mu \cdot \eta$ in $\Omega'^2_{Z^2(f)}$ for some $\mu \in pR$ called the eigenvalue.

b) Assume that moreover $\eta = d\nu$ for some canonical 1–form ν on $J^1(Y)$. Then we call

$$\epsilon := \frac{\phi_Z\nu - \mu\nu}{p} \in \Omega'_{Z^2(f)} \tag{4.8}$$

an Euler–Lagrange form.

We consider (4.8) as an (admittedly, half–baked) arithmetical analog of the expression $i_{\partial_\tau}(ydz - \mathcal{H}d\tau)$ (cf. (4.5)) in the same sense as the p–derivation

$$\delta_p(x) = \frac{\phi(x) - x^p}{p}$$

is an analog of ∂_τ.

Now we pass to the arithmetical PVI. Let again E be an elliptic curve over R that does not admit a lift of Frobenius and let $\psi \in \mathcal{O}(J^2(E))$ be the canonical δ-character of order 2 attached to an invertible 1–form ω on E. Consider the symplectic form $\eta = \omega^{(0)} \wedge \omega^{(1)}$ on $J^1(E)$: cf. (3.4). Let $Y \subset E$ be an affine open set and let $r \in \mathcal{O}(Y)$. Assume in addition that Y has an étale coordinate. (The basic example is E with sections of the second order deleted).

Denoting such an étale coordinate by T, put $\mathcal{A}_2 = K[[T, T']]$, $\mathcal{A}_3 = K[[T, T', T'']]$, where K is the quotient ring of R.

4.4. Proposition. *The following assertions hold:*

1) The function $f = \psi - \phi(r)$ defines a generalised canonical δ–flow on $J^1(Y)$ which is Hamiltonian with respect to η.

2) There exists a canonical 1-form ν on X such that $d\nu = \eta$; in particular the symplectic form η is exact and if $\epsilon := \frac{1}{p}(\phi_Z^*\nu - \mu\nu)$ is the Euler-Lagrange form then $p\epsilon$ is closed.

3) Let ϵ be Euler-Lagrange form and $f = \psi - \phi(r)$. Then we have the following equality in $\Omega_{\mathcal{A}_3}$:

$$\epsilon = f\omega^{(1)} - \frac{1}{p}(\phi^* - \mu)\nu.$$

4) Let $r_1, r_2 \in \mathcal{O}(Y)$ be such that $r_2 - r_1 = \partial s$, for some $s \in \mathcal{O}(Y)$, where ∂ is the derivation on E dual to ω. (This holds for two right hand sides of any two PVI equations). Consider the equations $\psi - \phi(r_1)$ and $\psi - \phi(r_2)$ respectively. Then there exists a canonical 1–form ν on $J^1(Y)$ such that $d\nu = \eta$ and such that, if ϵ_1, ϵ_2 are the corresponding Euler–Lagrange forms, then :

$$\epsilon_1 - \epsilon_2 = \frac{1}{p}d\phi(s) \in \Omega_{\mathcal{A}_2}.$$

Remark. We will deduce from this Proposition below (cf. Corollary 5.3.3) that in fact p–adic PVI in the form (3.6) defines a generalised canonical δ–flow. However, it does not define a δ–flow in the sense of Definition 1.3.4. This motivated our Definition 4.2.1.

Proof. From (3.4), we get the following equality in $\Omega^2_{J^2(Y)}$:

$$\frac{\phi^*\eta}{p^2} = \frac{\phi^*\omega^{(0)}}{p} \wedge \frac{\phi^*\omega^{(1)}}{p} = \omega^{(1)} \wedge \omega^{(2)}.$$

Recall the formula (3.9b):

$$df = p\omega^{(2)} + (\lambda_1 - p\phi(\partial r))\omega^{(1)} + \lambda_0\omega^{(0)}$$

in $\Omega_{J^2(E)}$ where $\lambda_1 \in R$, $\lambda_0 \in R^{\times}$. Hence, if we keep notation $\omega^{(0)}, \omega^{(1)}$ also for the images of the respective forms in $\Omega'_{Z^2(f)}$, in view of $\omega^{(1)} \wedge df = 0$ in $\Omega'_{Z^2(f)}$, we have the following equality in $\Omega'^2_{Z^2(f)}$:

$$\phi_Z^*\eta = -p \cdot \omega^{(1)} \wedge ((\lambda_1 - p\phi(dr))\omega^{(1)} + \lambda_0\omega^{(0)}) = p\lambda_0\eta,$$

This completes the proof of 1).

Write now $\omega = dL = \frac{dL}{dT}dT$ where $L = L(T) \in TK[[T]]$. (For instance, we can take L to be the formal logarithm of E). Then

$$\mathcal{A}_3 = K[[T, \phi(T), \phi^2(T)]] = K[[L, \phi(L), \phi^2(L)]].$$

So the image of ψ in $\mathcal{A}_3$ is

$$\psi = \frac{1}{p}(\phi^2(L) + \lambda_1\phi(L) + p\lambda_0 L) + \lambda_{-1}$$

for some $\lambda_{-1} \in R$. Therefore the maps

$$\Omega_{\mathcal{A}_2} \to \frac{\Omega_{\mathcal{A}_3}}{\langle f\Omega_{\mathcal{A}_3}, df\rangle}, \qquad \Omega^2_{\mathcal{A}_2} \to \frac{\Omega^2_{\mathcal{A}_3}}{\langle f\Omega^2_{\mathcal{A}_3}, df \wedge \Omega_{\mathcal{A}_3}\rangle}$$

are isomorphisms and so we have induced Frobenii maps $\phi_f^* : \Omega_{\mathcal{A}_2} \to \Omega_{\mathcal{A}_2}$ and $\phi_f^* : \Omega^2_{\mathcal{A}_2} \to \Omega^2_{\mathcal{A}_2}$. Since T is étale the lift of Frobenius $T \mapsto T^p$ on $\mathbf{A}^1$ extends to a lift of Frobenius $\phi_0 : \widehat{Y} \to \widehat{Y}$ of the p–adic completion of Y. Also the derivation $\frac{d}{dT}$ on $R[T]$ extends to a derivation still denoted by $\frac{d}{dT}$ on $\mathcal{O}(\widehat{Y})$. We claim that

$$\frac{\phi(L) - \phi_0(L)}{p},$$

which a priori is an element of $\mathcal{A}_2$, actually belongs to $\mathcal{O}(J^1(Y))$. Indeed we have the following expansion in $\mathcal{A}_2$:

$$\frac{\phi(L) - \phi_0(L)}{p} = \frac{L^{(\phi)}(T^p + pT') - L^{(\phi)}(T^p)}{p}$$

$$= \sum_{i=1}^{\infty} \frac{p^{i-1}}{i!} \frac{d^i L^{(\phi)}}{dT^i}(T^p)(T')^i = \sum_{i=1}^{\infty} \frac{p^{i-1}}{i!} \phi_0 \left(\frac{d^i L}{dT^i} \right) (T')^i$$

$$= \sum_{i=1}^{\infty} \frac{p^{i-1}}{i!} \phi_0 \left(\left(\frac{d}{dT} \right)^{i-1} \left(\frac{\omega}{dT} \right) \right) (T')^i \in \mathcal{O}(J^1(Y)),$$

where the superscript (ϕ) means twisting coefficients by ϕ. The latter inclusion follows because $T' = \delta T \in \mathcal{O}(J^1(Y))$, $\frac{\omega}{dT} \in \mathcal{O}(Y) \subset \mathcal{O}(\widehat{Y})$, and the latter is stable under $\frac{d}{dT}$ and ϕ_0. Now set

$$\nu := -\frac{\phi(L) - \phi_0(L)}{p} \, \omega \in \mathcal{O}(J^1(Y)) \, \omega \in \Omega_{J^1(Y)}.$$

Then

$$d\nu = -d \left(\frac{\phi(L) - \phi_0(L)}{p} \right) \wedge \omega$$

$$= -d \left(\frac{\phi(L)}{p} \right) \wedge \omega + d \left(\frac{\phi_0(L)}{p} \right) \wedge \omega$$

$$= -\omega^{(1)} \wedge \omega^{(0)} = \eta.$$

This completes the proof of 2).

Next for $f = \psi - \phi(r)$ we have the following computation in $\Omega_{\mathcal{A}_3}$:

$$p\epsilon = \phi_f^* \nu - \mu\nu$$

$$= -\phi_f^*\left(\frac{\phi(L)-\phi_0(L)}{p}\omega\right) + \mu\frac{\phi(L)-\phi_0(L)}{p}\omega$$

$$= -\phi_f\phi(L)\omega^{(1)} + \phi\phi_0(L)\omega^{(1)} + \mu\frac{\phi(L)-\phi_0(L)}{p}\omega$$

$$= (\lambda_1\phi(L) + p\lambda_0 L + \phi\phi_0(L) - p\phi(r) + p\lambda_{-1})\omega^{(1)} + \mu\frac{\phi(L)-\phi_0(L)}{p}\omega^{(0)}$$

$$= (\lambda_1\phi(L) + p\lambda_0 L + \phi\phi_0(L) - p\phi(r) + p\lambda_{-1})\omega^{(1)}$$

$$+\phi^*\left(\frac{\phi(L)-\phi_0(L)}{p}\omega^{(0)}\right) - (\phi^*-\mu)\nu$$

$$= \left(\phi^2(L) + \lambda_1\phi(L) + p\lambda_0 L - p\phi(r) + p\lambda_{-1}\right)\omega^{(1)} - (\phi^*-\mu)\nu$$

$$= pf\omega^{(1)} - (\phi^*-\mu)\nu.$$

This ends the proof of assertion 3). Assertion 4) follows from the fact that

$$\epsilon_1 - \epsilon_2 = \phi(\partial s)\omega^{(1)} = \frac{1}{p}d(\phi(s)).$$

Remarks. a) Some of our arguments above break down if $\psi - \phi(r)$ is replaced by $\psi - r$. This is our main motivation for studying $\psi - \phi(r)$.

b) Assertion 3) implies that if $[\epsilon]_2, [f\omega^{(1)}]_2 \in \frac{\Omega_{\mathcal{A}_3}}{(\phi^*-\mu)\Omega_{\mathcal{A}_2}}$ are the images of $\epsilon, f\omega^{(1)}$ then

$$[\epsilon]_2 = [f\omega^{(1)}]_2.$$

This justifies our suggestion that f is the "Euler–Lagrange equation attached to our Hamiltonian data". Assertion 4) says that the Euler–Lagrange forms of various PVI equations differ by exact forms.

c) Finally, we could treat in this way also the multicomponent version of PVI and the degeneration PV as they are described in [Ta].

4.5. Geometry of PVI in various categories. Below we essentially reproduce from [Ma96] a geometric description of the natural habitats of various forms of PVI including its arithmetic version.

Our series of constructions starts with a non–constant family ("pencil") of elliptic curves over one–dimensional base. in one of several natural categories: schemes over a field of characteristic zero or a ring of algebraic integers, or a p–adic completen of the latter, analytic spaces etc.

a. Let $(\pi : E \to B; D_0, \dots, D_3)$ be a pencil of compact smooth curves of genus one, with variable absolute invariant, endowed with four labelled sections D_i such that if any one of them is taken as zero, the others will be of order two.

We will call E *a configuration space* of PVI (common for all values of parameters.) Solutions to all equations will be represented by some multisections of π.

b. Let $\mathcal{F}$ be the subsheaf of the sheaf of vertical 1-forms $\Omega^1_{E/B}(D_3)$ on E with pole at D_3 and residue 1 at this pole. It is an affine twisted version of $\Omega^1_{E/B}$ which is the sheaf of sections of the relative cotangent bundle $T^*_{E/B}$. Similarly, $\mathcal{F}$ itself "is" the sheaf of sections of an affine line bundle $F = F_{E/B}$ on E. More precisely, we can construct a bundle $\lambda : F \to E$ and a form $\nu_F \in \Gamma(F, \Omega^1_{F/B}(\lambda^{-1}(D_3))$ such that the map

$$\{\text{local section } s \text{ of } F\} \mapsto s^*(\nu_F)$$

identifies the sheaf of sections of F/E with $\mathcal{F}$.

We will call F *a phase space* for PVI (again, common for all parameter values.) It is this space that carries a canonical symplectic form (relative over the base) rather than the usual cotangent bundle.

c. E carries a distinguished family of algebraic/arithmetic curves transversal to the fibers of E: considered as multisections of E/B they are of finite order (if any of D_i is chosen as zero.) It is important that each curve of this family has a canonical lifting to F (for its description, see [Ma96], especially (2.12) and (2.29).)

d. F carries a closed 2-form ω which can be characterised by the following two properties:

i). The vertical part of ω, i. e. its restriction to the fibers of $\pi \circ \lambda : F \to B$, coincides with $d_{F/B}(\nu_F)$.

ii). Any canonical lift to F of a connected multisection of finite order of $E \to B$, referred to above, is a leaf of the null–foliation of ω.

e. E also carries four distinguished closed two–forms $\omega_0, \dots, \omega_3$. They are determined, up to multiplication by a constant, by the following properties.

iii). The divisor of ω_i *is* $\dfrac{D_jD_kD_l}{D_i^3}$ *where* $\{i,j,k,l\}=\{0,1,2,3\}$.

iv). Identify the sheaves Ω^2_E *and* $\pi^*(\Omega^1_{E/B})^{\otimes 3}$ *on* E *using the Kodaira–Spencer isomorphism* $\pi^*(\Omega^1_B)\cong(\Omega^1_{E/B})^{\otimes 2}$ *and the exact sequence* $0\to\pi^*(\Omega^1_B)\to\Omega^1_E\to\Omega^1_{E/B}\to 0$. *Then the image of* ω_i *in* $\pi^*(\Omega^1_{E/B})^{\otimes 3}$ *considered in the formal neighborhood of* D_i *is the cube of a vertical 1–form with a constant residue along* D_i.

Notice that up to introduction of ω, all constructions were valid both in geometric and arithmetic cases. It is precisely the absence of the differential in arithmetic direction was the object of our concerns in 4.2–4.4 above.

The affine space $P_0:=\omega+\sum_{i=0}^3\mathbf{C}\lambda^*(\omega_i)$ of closed two–forms on F is our version of the *moduli space of the PVI equations* replacing the classical $(\alpha,\beta,\gamma,\delta)$–space.

We can now summarize our geometric definition of PVI equations and their solutions.

4.5.1. Definition *a) A Painlevé two–form on* F *is a point* $\Omega\in P_0$.

b) The Painlevé foliation corresponding to Ω *is the null–foliation of* Ω.

c) The solutions to the respective Painlevé equation are the leaves of this foliation (in the Hamiltonian description).

The form ω *corresponds to* $(\alpha_0,\alpha_1,\alpha_2,\alpha_3)=(0,0,0,0)$.

5. Pfaffian congruences

5.1. Notation. We will continue here the discussion started in 3.3.

For $u\in R$, define the following infinite vectors in $R\times R\times\dots$:

$$\nabla(u):=(u,\delta u,\delta^2u,...),$$

$$\delta\nabla(u):=(\delta u,\delta^2u,\delta^3u,...),$$

$$\nabla(u)^p:=(u^p,(\delta u)^p,(\delta^2u)^p,...).$$

For $g\in R[y,y',...,y^{(r)}]\widehat{\ }$, denote by $g^{(\phi)}$ be the series obtained from g by applying ϕ to the coefficients of g. Moreover, put

$$dg:=\sum_{i=0}^r\frac{\partial g}{\partial y^{(i)}}dy^{(i)},$$

$$\frac{\partial g}{\partial p} := \frac{1}{p}(g^{(\phi)}(y^p, \dots (y^{(r)})^p) - g(y, \dots, y^{(r)})^p),$$
$$\frac{\partial}{\partial \nabla(y)} := \left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial y''} \dots \right),$$

Denoting by $\langle\, , \rangle$ the basic pairing between 1–forms and vector fields, and writing $\left(\frac{\partial}{\partial \nabla(y)}\right)^t$ for the column transpose of $\frac{\partial}{\partial \nabla(y)}$, we get for any $f \in R[y, y', ..., y^{(r)}]\widehat{\ }$ and $i \geq 0$:

$$\delta^{i+1} f(u) = \delta\delta^i f(u)$$
$$= \frac{1}{p}(((\delta^i f)^{(\phi)}(u^p + p\delta u, (\delta u)^p + p\delta^2 u, \dots) - (\delta^i f)(u, \delta u, \dots)^p)$$
$$\equiv \frac{\partial \delta^i f}{\partial p}(\nabla(u)^p) + \sum_{j=0}^{r} \frac{\partial (\delta^i f)^{(\phi)}}{\partial y^{(j)}}(\nabla(u)^p)\delta^{j+1} u \bmod p$$
$$\equiv \frac{\partial \delta^i f}{\partial p}(\nabla(u)^p) + \langle\, d[(\delta^i f)^{(\phi)}](\nabla(u)^p), \delta\nabla(u) \cdot \left(\frac{\partial}{\partial \nabla(y)}\right)^t \rangle \bmod p.$$

It is known ([Bu05], Lemma 3.20), that if $b \in R$ then $b = 0$ if and only if

$$\delta^i b \equiv 0 \bmod p \quad \text{for all } i \geq 0.$$

Combining the above facts we get:

5.2. Proposition. *An element $u \in R$ is a solution to $f(u, \delta u, ..., \delta^r u) = 0$ if and only if the following hold:*

a) $f(\nabla(u)) \equiv 0 \bmod p$,

b) $\frac{\partial (\delta^i f)}{\partial p}(\nabla(u)^p) \equiv -\langle\, d[(\delta^i f)^{(\phi)}](\nabla(u)^p), \delta\nabla(u) \cdot \left(\frac{\partial}{\partial \nabla(y)}\right)^t \rangle \bmod p, \quad i \geq 0.$

Moreover, u is a solution to the equation $\delta f(u, \delta u, ..., \delta^r u) = 0$ if and only if b) above holds.

The above discussion can be obviously generalized to the case when y is a tuple of variables and f is a tuple of equations. It shows that the equation $f(u, \delta u, ..., \delta^r u) = 0$ is controlled by a system of "Pfaffian" congruences involving the 1-forms

$$d((\delta^i f)^{(\phi)}) = d(\delta^i (f^{(\phi)})), \quad i \geq 0.$$

If f has $\mathbf{Z}_p$-coefficients then the above forms are, of course,

$$d(\delta^i f), \quad i \geq 0.$$

In what follows we analyse such forms relevant for PVI equations.

5.3. The forms $\delta^i(\psi - \rho)$. We start by defining inductively certain universal δ-polynomials.

Consider a family of commuting free variables $z = (z_0, z_1, \ldots, z_r)$ and let w be another variable. As in the jet theory in [Bu05], denote by $z', z'', \ldots$, resp. $w', w'', ...$ new (families of) independent variables indexed additionally by the formal order of derivative. Consider the ring $\mathbf{Z}[z, z', z'', \ldots, w, w', w'', \ldots]$, equipped with the tautological p–derivation $\delta z_i^{(k)} = z_i^{(k+1)}$ etc., and hence with the lift of Frobenius $\phi(F) = F^p + p\delta F$. Define the elements $A_{m,i} = A_{m,i}(z, z', z'', \ldots, w, w', w' \ldots)$ of this ring ($m \geq 0$, $i = 0, ..., m + r$) by induction:

$$\begin{aligned} A_{0,i} &:= z^{(i)}, \quad i = 0, ..., r \\ A_{m+1,0} &:= -(w^{(m)})^{p-1} A_{m,0}, \quad m \geq 0 \\ A_{m+1,i} &:= \phi(A_{m,i-1}) - (w^{(m)})^{p-1} A_{m,i}, \ i = 1, ..., m + r, \quad m \geq 0 \\ A_{m+1,m+r+1} &:= \phi(A_{m,m+r}), \ m \geq 0. \end{aligned} \tag{5.1}$$

It is easy then to check that

$$\begin{aligned} A_{m,0} &= (-1)^{m-1}(ww'...w^{(m-1)})^{p-1} z^{(0)}, \quad m \geq 1, \\ A_{m,m+r} &= \phi^m(z^{(r)}), \quad m \geq 0, \\ A_{m,m+r-1} &\equiv \phi^m(z^{(r-1)}) \bmod (z^{(r)}, \phi(z^{(r)}), \ldots, \phi^{m-1}(z^{(r)})), \quad m \geq 0 \\ A_{m,m+i} &\equiv \phi^m(z^{(i)}) \bmod (w, w', \ldots, w^{(m-1)}), \quad m \geq 1, \quad i = 0, \ldots, r, \\ A_{m,i} &\equiv 0 \bmod (w, w', \ldots, w^{(m-1)}), \quad i = 0, \ldots, m - 1. \end{aligned} \tag{5.2}$$

In the following statement Y is an affine smooth curve over R and $f \in \mathcal{O}(J^r(Y))$. Then by [Bu05], there exist $a_0, ..., a_r \in \mathcal{O}(J^r(Y))$ such that

$$df = \sum_{i=0}^{r} a_i \omega^{(i)}.$$

Put also $a = (a_1, \dots, a_r)$. Then we have:

5.3.1. Proposition. *Put* $a_{m,i} = A_{m,i}(a, \delta a, \delta^2 a, \dots, f, \delta f, \delta^2 f, ...)$. *Then*

$$d(\delta^m f) = \sum_{i=0}^{m+r} a_{m,i}\omega^{(i)}.$$

Proof. We proceed by induction on $m \geq 0$. The case $m = 0$ is trivial. The passage from m to $m+1$ runs as follows:

$$d(\delta^{m+1} f) = d(\delta\delta^m f) = d\left(\frac{\phi(\delta^m f) - (\delta^m f)^p}{p}\right) = \frac{\phi^*}{p}(d(\delta^m f)) - (\delta^m f)^{p-1} d(\delta^m f)$$

$$= \frac{\phi^*}{p}(\sum_{i=0}^{m+r} a_{m,i}\omega^{(i)}) - (\delta^m f)^{p-1}(\sum_{i=0}^{m+r} a_{m,i}\omega^{(i)})$$

$$= (\sum_{i=0}^{m+r} \phi(a_{m,i})\omega^{(i+1)}) - (\delta^m f)^{p-1}(\sum_{i=0}^{m+r} a_{m,i}\omega^{(i)}) = \sum_{i=0}^{m+1+r} a_{m+1,i}\omega^{(i)}.$$

This ends the proof.

Now we apply this to the case when Y is an affine open subset of the elliptic curve E over $\mathbf{Z}_p$ without lift of Frobenius. Denote by $a_p \in \mathbf{Z}$ the trace of Frobenius on the reduction $E \bmod p$.

5.3.2. Corollary. *Let* $f = \psi - \rho$, $\rho = \phi(r)$, *where* $r \in \mathcal{O}(Y)$, ψ *the canonical* δ*-character of order* 2 *attached to* ω. *Let* $d(\delta^m f) = \sum_{i=0}^{m+2} a_{m,i}\omega^{(i)}$, $m \geq 0$, $a_{m,i} \in \mathcal{O}(J^{m+2}(Y))$. *So*

$$a_{0,2} = p,\ a_{0,1} = -(a_p + p\phi(\frac{dr}{\omega})),\ a_{0,0} = 1.$$

Then, for $m \geq 1$*:*

1) $a_{m,m+2} = p$;
2) $a_{m,m+1} \equiv -a_p \bmod p$;
3) $a_{m,m+1} \equiv -(a_p + p\phi^{m+1}(\frac{dr}{\omega})) \bmod (f, \delta f, \dots, \delta^{m-1} f)$;
4) $a_{m,m} \equiv 1 \bmod (f, \delta f, \dots, \delta^{m-1} f)$;
5) $a_{m,i} \equiv 0 \bmod (f, \delta f, \dots, \delta^{m-1} f)$ *for* $i = 0, \dots, m-1$;

6) $a_{m,0} = (-1)^{m-1}(f \cdot \delta f \cdots \delta^{m-1} f)^{p-1}$*, for* $m \geq 1$*.*

In particular if E *has ordinary reduction then* $a_{m,m+1}$ *is invertible in* $\mathcal{O}(J^{m+2}(Y))$*.*

One can prove a similar statement for $\psi - r$ in place of $\psi - \phi(r)$.

5.3.3. Corollary. *In the same situation, let* $f = \psi - \rho$*, where* $\rho = r$ *or* $\rho = \phi(r)$*. Let* ψ *be the canonical* δ*-character of order* 2 *attached to* ω*.*

Denote by $Z^{m+2} = Z^{m+2}(f, \delta f, ..., \delta^m f)$ *the closed formal subscheme of* $J^{m+2}(Y)$ *defined by the ideal generated by* f*,* δf*, ...,* $\delta^m f$*. Let* $\pi_m : Z^{m+2} \to J^1(Y)$ *be the canonical projection.*

Then the map

$$\pi_m^* \Omega_{J^1(Y)} \to \Omega'_{Z^{m+2}}$$

is injective with cokernel annihilated by p^{m+1}*. In particular, for any* $m \geq 0$*, the system* $\{f, \delta f, ..., \delta^m f\} \subset \mathcal{O}(J^{m+2}(Y))$ *defines a generalised* δ*–flow on* $J^1(Y)$*. If moreover* E *has ordinary reduction then the above cokernel is a cyclic* $\mathcal{O}(Z^{m+2})$*-module generated by the class of* $\omega^{(m+2)}$*.*

Proof. Consider the case $\rho = \phi(r)$; a similar argument holds for $\rho = r$. Recall that $\Omega_{J^n(Y)}$ is a free $\mathcal{O}(J^n(Y))$-module generated by $\omega^{(0)}, ..., \omega^{(n)}$. Also, by definition,

$$\mathcal{O}(Z^{m+2}) = \frac{\mathcal{O}(J^{m+2}(Y))}{(f, \delta f, ..., \delta^m f)}$$

and

$$\Omega'_{Z^{m+2}} = \frac{\mathcal{O}(J^{m+2}(Y))\omega^{(0)} \oplus ... \oplus \mathcal{O}(J^{m+2}(Y))\omega^{(m+2)}}{\langle (\delta^i f)\omega^{(j)}, d(\delta^i f) \rangle},$$

where $\langle\ \ \rangle$ means $\mathcal{O}(J^{m+2}(Y))$-linear span and $j = 0, ..., m+2$, $i = 0, ..., m$. By Corollary 5.3.2 we have:

$$\Omega'_{Z^{m+2}} = \frac{\mathcal{O}(Z^{m+2})\omega^{(0)} \oplus ... \oplus \mathcal{O}(Z^{m+2})\omega^{(m+2)}}{\langle p\omega^{(2+i)} - (a_p + p\phi^{i+1}(\frac{dr}{\omega}))\omega^{(1+i)} + \omega^{(i)} \rangle}$$

where $i = 0, ..., m$ and $\langle\ \ \rangle$ means here $\mathcal{O}(Z^{m+2})$-linear span. Note that $\pi_m^* \Omega_{J^1(Y)}$ is a free $\mathcal{O}(J^{m+2}(Y))$-module with basis $\omega^{(0)}$, $\omega^{(1)}$. So in order to prove that the map

$$\pi_m^* \Omega_{J^1(Y)} \to \Omega'_{Z^{m+2}} \tag{5.3}$$

is injective we need to check that no $\mathcal{O}(Z^{m+2})$-linear combination of $\omega^{(0)}$, $\omega^{(1)}$ can be a $\mathcal{O}(Z^{m+2})$-linear combination of elements $p\omega^{(2+i)}-(a_p+p\phi^{i+1}(\frac{dr}{\omega}))\omega^{(1+i)}+\omega^{(i)}$ which is clear. For $j=2,...,m+2$ let

$$\overline{\omega^{(j)}} \in \frac{\Omega'_{Z^{m+2}}}{\langle \omega^{(0)}, \omega^{(1)} \rangle}$$

be the image of $\omega^{(j)}$. Clearly $p\overline{\omega^{(2)}}=0$ hence $p^2\overline{\omega^{(3)}}=0$, etc. We conclude that p^{m+1} annihilates the cokernel of (5.3).

Finally assume E has ordinary reduction. We want to show that the above cokernel is generated by $\overline{\omega^{(m+2)}}$. It is enough to show that for all $j=1,...,m+2$ we have

$$\overline{\omega^{(j-1)}} \in p\,\mathcal{O}(Z^{m+2})\,\overline{\omega^{(j)}}.$$

We proceed by induction on j. For $j=1$ this is clear. Now assume the above is true for some $1 \leq j < m+2$. We have

$$p\overline{\omega^{(j+1)}} - (a_p + p\phi^j(\frac{dr}{\omega}))\overline{\omega^{(j)}} + \overline{\omega^{(j-1)}} = 0.$$

By induction $\overline{\omega^{(j-1)}} = p\,c\,\overline{\omega^{(j)}}$ for some $c \in \mathcal{O}(Z^{m+2})$. Since a_p is invertible in R it follows that $a_p - pc(1+\phi^{j-1}(dr/\omega))$ is invertible hence

$$\overline{\omega^{(j)}} = p(a_p + p\phi^j(\frac{dr}{\omega}) - pc)^{-1}\overline{\omega^{(j+1)}},$$

which ends the proof.

5.4. Remark. Let Y be a smooth affine scheme over R and let $\mathcal{F} \subset \mathcal{O}(J^r(Y))$ be a system of arithmetic differential equations. Consider the set

$$\{\mathcal{F}, \delta\mathcal{F}, \ldots, \mathcal{F}\} \subset \mathcal{O}(J^{r+m}(Y))$$

which can be referred to as the m–th prolongation of $\mathcal{F}$. Let

$$Z^{r+m} := Z^{r+m}(\mathcal{F}, \delta\mathcal{F}, \ldots, \delta^m\mathcal{F}) \subset J^{r+m}(Y)$$

be the closed subscheme defined by this prolongation and let

$$Z^{\infty} = Z^{\infty}(\delta^{\infty}\mathcal{F})$$

be defined by

$$Z^\infty = \lim_{\leftarrow} Z^{r+m}$$

be the projective limit (defined as the Spf of the p–adic completion of the inductive limit of $\mathcal{O}(Z^{r+m})$ as m varies; this is a generally non-Noetherian formal scheme). Then Z^∞ is a closed horizontal formal subscheme of

$$J^\infty(Y) := \lim_{\leftarrow} J^n(Y);$$

by horizontal we mean here that its ideal is sent into itself by δ. So there is an induced p–derivation δ_{Z^∞} on $\mathcal{O}(Z^\infty)$.

If Z^∞ happens to be a smooth formal scheme then δ_{Z^∞} defines a (genuine) δ-flow $\mathcal{E}(\delta_{Z^\infty}) \subset \mathcal{O}(J^1(Z^\infty))$ on Z^∞ (in the sense of our previous definition). In case Z^∞ is not necessarily a smooth formal scheme one can still define the affine formal scheme $J^1(Z^\infty)$ and the δ–flow $\mathcal{E}(\delta_{Z^\infty})$ on Z^∞ by copying the definitions from the smooth case.

The following problem needs to be investigated. Assume that $\mathcal{F} \subset \mathcal{O}(J^2(Y))$ defines a δ-flow $\mathcal{E}(\delta_{J^1(Y)}) \subset \mathcal{O}(J^1(J^1(Y)))$ on $J^1(Y)$. Consider the formal scheme $Z^\infty = Z^\infty(\delta^\infty \mathcal{F})$ as above. Prove that Z^∞ is naturally isomorphic to $J^1(Y)$ and $\mathcal{E}(\delta_{J^1(Y)})$ is naturally identified with $\mathcal{E}(\delta_{Z^\infty})$. This would show the naturality of the above definitions.

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